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INVERSE DOUBLY PERIODIC PROBLEM OF THE THEORY OF BENDING OF A PLATE WITH ELASTIC INCLUSIONS

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Based on the balanced strength principle, a problem of determining the optimal interference for fitting elastic inclusions into holes of an isotropic elastic plate weakened by a doubly periodic system of circular holes is solved. A closed system of algebraic equations is derived, which allows solving this problem. The resultant interference increases the load-carrying capacity of the composite plate being bent.

Key words: *foreign inclusions, perforated plate, fitting interference, bending, optimal design.*

Introduction. The practice shows that multicomponent structures are more reliable and have a longer lifetime than homogeneous structures [1]. In designing composite materials, the load-carrying capacity of a perforated plate with circular holes can be improved by reinforcing the contours of these holes with interference by elastic disks made of another elastic material. These reinforcing elements have a moderate mass but significantly affect the plate strength. The operational lifetime of a composite (multicomponent) structure depends on the distribution of stresses in zones of interaction of its elements; therefore, optimal design of such structures, i.e., determining their optimal characteristics, becomes extremely important. The workability of a composite plate can be improved by design and technological methods, in particular, by changing the geometry (interference) of connection of its elements. Similar problems of mechanics were solved in [2–9].

Formulation of the Problem. Let us consider an isotropic elastic plate weakened by a doubly periodic system of circular holes of radius λ ($\lambda < 1$) whose centers are located at the points $P_{mn} = m\omega_1 + n\omega_2$ ($m, n = 0, \pm 1, \pm 2, \ldots$), where $\omega_1 = 2$, $\omega_2 = 2h_* \exp(i\alpha)$, $h_* > 0$, and $\operatorname{Im}\omega_2 > 0$.

We have to determine the interference for fitting the inclusions into the holes on such a plate. It should be noted that no solutions are available for problems of the elasticity theory on constructing a system of concentrators (inclusions) such that the elastic field induced by this system could reduce the stress concentration in the perforated plate. The plate is subjected to uniform bending by uniformly distributed constant moments (bending at infinity)

$$M_x = M_x^\infty, \qquad M_y = M_y^\infty, \qquad H_{xy} = 0.$$

The origin of the coordinate system is placed into the geometric center of the hole $L_{0,0}$ in the mid-plane $x0y$ of the plate.

We assume that elastic disks made of a different elastic material are fitted with interference into the circular holes of the plate L_{mn} ($m, n = 0, \pm 1, \pm 2, \ldots$) by means of press fitting or thermal influence. The disks have a greater size than the plate holes, and the disk thickness equals the plate thickness. Owing to the symmetry of the boundary conditions and the geometry of the region occupied by the elastic medium, the stresses in the plate being bent are doubly periodic functions with the fundamental periods ω_1 and ω_2.

The complex potentials that refer to the disk are denoted by $\Phi_0(z)$ and $\Psi_0(z)$, and those that refer to the plate are indicated by $\Phi(z)$ and $\Psi(z)$. As the solution for the plate possesses the property of double periodicity, it suffices to consider the conditions of plate–inclusion junction only along the contour of the basic hole $L_{0,0}$.

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The boundary conditions for the problem considered have the following form [10]:

$$\Phi(\tau) + \overline{\Phi(\tau)} - [\bar{\tau}\Phi'(\tau) + \Psi(\tau)]\,\mathrm{e}^{2i\theta} = \Phi_0(\tau) + \overline{\Phi_0(\tau)} - [\bar{\tau}\Phi_0'(\tau) + \Psi_0(\tau)]\,\mathrm{e}^{2i\theta} + g'(\tau); \quad (1)$$

$$\varepsilon\overline{\Phi(\tau)} + \Phi(\tau) - [\bar{\tau}\Phi'(\tau) + \Psi(\tau)]\,\mathrm{e}^{2i\theta} = \frac{D_0(1-\nu_0)}{D(1-\nu)}\,\{\varepsilon_0\overline{\Phi_0(\tau)} + \Phi_0(\tau) - [\bar{\tau}\Phi_0'(\tau) + \Psi_0(\tau)]\,\mathrm{e}^{2i\theta}\}. \quad (2)$$

Here $\tau = \lambda\,\mathrm{e}^{i\theta} + m\omega_1 + n\omega_2$ $(m, n = 0, \pm 1, \pm 2, \ldots)$, ν and ν_0 are Poisson's ratio of the plate and disk materials, respectively, D and D_0 are the cylindrical rigidities of the plate and the disk, respectively, $g(\tau)$ is the sought interference function to be determined from an additional condition, $\varepsilon = -(3+\nu)/(1-\nu)$, and $\varepsilon_0 = -(3+\nu_0)/(1-\nu_0)$.

The sought complex function $g(\tau)$ characterizes the jumps in displacements at the interface between the media:

$$(u^+ - u^-) + i(v^+ - v^-) = g(\tau) \qquad \text{on } L_{mn}.$$

The function $g(\tau)$ depends on the geometry of the inserted disks before their deformation and on the method used to bring the points belonging to the disk and hole contours into contact.

According to the Kirchhoff theory, the problem considered reduces to finding two pairs of functions $\Phi_0(z)$, $\Psi_0(z)$, $\Phi(z)$, and $\Psi(z)$ of the complex variable $z = x + iy$, which are analytical in the corresponding domains and satisfy the boundary conditions (1) and (2).

To find the interference function, we use the condition of balanced strength on the contours of the circular holes as a condition for determining the fitting interference [function $g(\theta)$]. We have to determine the function $g(\theta)$ such that the stress–strain field generated by interference in the course of loading of a composite body should provide balanced strength on the contours of the circular holes in the bent plate. This additional condition allows us to find the sought function $g(\theta)$ of press fitting interference.

Method of the Solution. We seek for the complex potentials $\Phi_0(z)$ and $\Psi_0(z)$, which describe the stress–strain state of the disk in the following form [11]:

$$\Phi_0(z) = \sum_{k=0}^{\infty} a_{2k} z^{2k}, \qquad \Psi_0(z) = \sum_{k=0}^{\infty} a'_{2k} z^{2k}. \quad (3)$$

The complex potentials $\Phi(z)$ and $\Psi(z)$, which describe the stress–strain state of the plate weakened by a doubly periodic system of circular holes, are sought with allowance for the mean moments as [10]

$$\Phi(z) = -\frac{M_x^\infty + M_y^\infty}{4D(1+\nu)} + \Phi_1(z), \qquad \Psi(z) = \frac{M_y^\infty - M_x^\infty}{2D(1-\nu)} + \Psi_1(z),$$

$$\Phi_1(z) = \alpha_0 + \sum_{k=0}^{\infty} \alpha_{2k+2}\,\frac{\lambda^{2k+2}\gamma^{(2k)}(z)}{(2k+1)!}, \quad (4)$$

$$\Psi_1(z) = \beta_0 + \sum_{k=0}^{\infty} \beta_{2k+2}\,\frac{\lambda^{2k+2}\gamma^{(2k)}(z)}{(2k+1)!} - \sum_{k=0}^{\infty} \alpha_{2k+2}\,\frac{\lambda^{2k+2}Q^{(2k+1)}(z)}{(2k+1)!},$$

where $\gamma(z)$ is the elliptical Weierstrass function and $Q(z)$ is a special meromorphic function (see [10]).

We give the dependences [10] that should be satisfied by the coefficients in Eqs. (4). As the basic vector and the basic moment of forces acting on the arc connecting two congruent points in the domain occupied by the plate material have zero values, we obtain

$$\alpha_0 = (K_0\alpha_2 + K_1\beta_2)\lambda^2, \qquad \beta_0 = (K_2\alpha_2 + K_3\beta_2)\lambda^2.$$

The expressions for the quantities K_i $(i = 0, 1, 2, 3)$ can be found in [10].

From the conditions of symmetry about the coordinate axes, we find

$$\operatorname{Im}\alpha_{2k} = 0, \qquad \operatorname{Im}\beta_{2k} = 0 \qquad (k = 0, 1, 2, \ldots).$$

We can easily verify that presentations (4) determine the class of symmetric problems with a doubly periodic distribution of stresses.

Without decreasing generality of the optimization problem posed, the sought function $g'(\tau)$ can be presented as a segment of the Fourier series

$$g'(\tau) = \sum_{k=-\infty}^{\infty} A_{2k}^{\text{int}}\, \mathrm{e}^{i2k\theta},$$

where

$$A_{2k}^{\text{int}} = \frac{1}{2\pi}\int_0^{2\pi} g'(\tau)\,\mathrm{e}^{-2ki\theta}\, d\theta, \qquad \operatorname{Im} A_{2k}^{\text{int}} = 0 \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

In this case, the optimization problem reduces to determining the coefficients A_{2k}^{int} $(k = 0, \pm 1, \pm 2, \ldots)$, which are control parameters.

We denote the left side of the boundary condition (1) by $f_1 - if_2$ and assume that the function $f_1 - if_2$ on the contour $L_{0,0}$ is expanded into a Fourier series. By virtue of symmetry, this series has the form

$$f_1 - if_2 = \sum_{k=-\infty}^{\infty} A_{2k}\,\mathrm{e}^{2ki\theta}, \qquad \operatorname{Im} A_{2k} = 0 \qquad (k = 0, \pm 1, \pm 2, \ldots). \tag{5}$$

Using the boundary condition (1) and relations (3) and (5) and applying the method of power series, we obtain the relations

$$a_0 = \frac{A_0 - A_0^{\text{int}}}{2}, \qquad a_{2k} = \frac{A_{-2k} - A_{-2k}^{\text{int}}}{\lambda^{2k}} \qquad (k = 1, 2, \ldots),$$

$$a'_{2k} = -(2k+1)\,\frac{A_{-2k-2} - A_{-2k-2}^{\text{int}}}{\lambda^{2k}} - \frac{A_{2k+2} - A_{2k+2}^{\text{int}}}{\lambda^{2k}} \qquad (k = 0, 1, 2, \ldots),$$

determining the coefficients a_{2k} and a'_{2k} of the functions $\Phi_0(z)$ and $\Psi_0(z)$.

To determine the values of A_{2k}, we consider the solution of the problem for the plate.

Using the complex potentials $\Phi_0(z)$ and $\Psi_0(z)$, we can write the boundary conditions on the contour of the circular hole ($\tau = \lambda\,\mathrm{e}^{i\theta}$) for the potentials $\Phi(z)$ and $\Psi(z)$ after some transformations as

$$\Phi(\tau) + \overline{\Phi(\tau)} - [\bar{\tau}\Phi'(\tau) + \Psi(\tau)]\,\mathrm{e}^{2i\theta} = \sum_{k=-\infty}^{\infty} A_{2k}\,\mathrm{e}^{2ki\theta}; \tag{6}$$

$$\varepsilon\overline{\Phi(\tau)} + \Phi(\tau) - [\bar{\tau}\Phi'(\tau) + \Psi(\tau)]\,\mathrm{e}^{2i\theta} = = \frac{D_0(1-\nu_0)}{D(1-\nu)}\Big(\frac{1+\varepsilon_0}{2}\,C_0 + \sum_{k=1}^{\infty} C_{2k}\,\mathrm{e}^{2ki\theta} + \varepsilon_0 \sum_{k=1}^{\infty} C_{-2k}\,\mathrm{e}^{-2ki\theta}\Big). \tag{7}$$

Here $C_{2k} = A_{2k} - A_{2k}^{\text{int}}$ $(k = 0, \pm 1, \pm 2, \ldots)$.

The boundary condition (6) serves to determine the coefficients α_{2k} and β_{2k}, and the boundary condition (7) is used to determine A_{2k}.

Using the methods described in [10], we obtain three infinite systems of linear algebraic equations with respect to α_{2k}, β_{2k}, and A_{2k}:

$$\alpha_{2j+2} = \sum_{j=1}^{\infty} a_{j,k}\alpha_{2k+2} + b_j \quad (j = 0, 1, 2, \ldots), \qquad a_{j,k} = (2j+1)\gamma_{j,k}\lambda^{2j+2k+2}; \tag{8}$$

$$\gamma^*_{0,0} = \frac{3}{8}\,g_2\lambda^2 + K_2 + \frac{(1+\varepsilon)\lambda^2 K_0 K_3}{K_4} + \varepsilon\sum_{i=1}^{\infty}\frac{(2i+1)g_{i+1}^2\lambda^{4i+2}}{2^{4i+4}},$$

$$\gamma^*_{0,k} = -\frac{(2k+2)\rho_{k+1}}{2^{2k+2}} + \frac{(2k+4)!g_{k+2}\lambda^2}{2!(2k+2)!2^{2k+4}} + \frac{(1+\varepsilon)\lambda^2 K_3 g_{k+1}}{K_4 2^{2k+2}} + \varepsilon\sum_{i=1}^{\infty}\frac{(2j+2i+1)!g_{i+1}g_{k+i+1}\lambda^{4i+2}}{(2k+1)!(2i)!2^{2k+4i+4}} \qquad (k = 1, 2, \ldots),$$

$$\gamma_{j,0} = -\frac{(2j+2)!\rho_{j+1}}{2^{2j+2}} + \frac{(2j+4)!g_{j+2}\lambda^2}{(2)!(2j+2)!2^{2j+4}} + \frac{(1+\varepsilon)\lambda^2 K_0 g_{j+1}}{[1-(1+\varepsilon)K_1\lambda^2]2^{2j+2}} \tag{9}$$

$$+ \varepsilon\sum_{i=1}^{\infty}\frac{(2j+2i+1)!g_{i+1}g_{j+i+1}\lambda^{4i+2}}{(2j+1)!(2i)!2^{4j+4i+4}} \qquad (j = 1, 2, \ldots),$$

$$\gamma_{j,k} = \gamma_{k,j} = -\frac{(2j+2k+2)!\rho_{j+k+1}}{(2j+1)!(2k+1)!2^{2j+2k+2}} + \frac{(2j+2k+4)!g_{j+k+2}\lambda^2}{(2j+2)!(2k+2)!2^{2j+2k+4}} + \frac{g_{j+1}g_{k+1}\lambda^2}{2^{2j+2k+4}}\Big[1 + \frac{(1+\varepsilon)^2K_1\lambda^2}{1-(1+\varepsilon)K_1\lambda^2}\Big]$$

$$+ \varepsilon\sum_{i=0}^{\infty}\frac{(2j+2i+1)!(2k+2i+1)!g_{j+i+1}g_{k+i+1}\lambda^{4i+2}}{(2j+1)!(2k+1)!(2i+1)!(2i)!2^{2j+2k+4i+4}} \quad (j,k = 1,2,\ldots).$$

The quantities $\gamma_{j,k}$ entering system (8) are determined by Eqs. (9) with $\varepsilon = 1$. The values of $\gamma^*_{j,k}$ are found by Eqs. (9) with $\varepsilon = -(3+\nu)/(1-\nu)$. The constants β_{2k} are determined from the following relations:

$$\beta_2 = \frac{1}{1-2K_1\lambda^2}\Big(-A_0 + \frac{M_x^\infty + M_y^\infty}{2D(1+\nu)} + 2\lambda^2\alpha_2K_0 + 2\sum_{k=1}^{\infty}\frac{g_{k+1}\lambda^{2k+2}\alpha_{2k+2}}{2^{2k+2}}\Big),$$

$$\beta_{2j+4} = (2j+3)\alpha_{2j+2} + \sum_{k=0}^{\infty}\frac{(2j+2k+3)!g_{j+k+2}\lambda^{2j+2k+4}\alpha_{2k+2}}{(2j+2)!(2k+1)!2^{2j+2k+4}} - A_{-2j-2}; \tag{10}$$

$$A_{2j+2} = \frac{1-\varepsilon}{1-\mu_0/\mu}\,\alpha_{2j+2} - \frac{\mu_0}{\mu(1-\mu_0/\mu)}\,A^{\text{int}}_{2j+2},$$

$$A_{-2j} = \frac{1-\varepsilon}{1-\varepsilon_0\mu_0/\mu}\sum_{k=0}^{\infty}r_{j,k}\lambda^{2k+2j+2}\alpha_{2k+2} - \frac{\varepsilon_0\mu_0}{\mu(1-\varepsilon_0\mu_0/\mu)}\,A^{\text{int}}_{-2j} \qquad (j = 0,1,2,\ldots),$$

$$A_0 = \sum_{k=0}^{\infty}Q_{0,k}\lambda^{2k+2}A_{2k+2} + Q_0\,\frac{M_x^\infty + M_y^\infty}{4D(1+\nu)} - \sum_{k=0}^{\infty}Q'_{0,k}\lambda^{2k+2}A^{\text{int}}_{2k+2} - \frac{(1+\varepsilon_0)\mu_0/\mu}{(1-2K_1\lambda^2)Q}\,A^{\text{int}}_0;$$

$$A_{2j} = \sum_{k=0}^{\infty}d_{j,k}A_{2k+2} + T_j \qquad (j = 0,1,2,\ldots), \tag{11}$$

where

$$Q_{0,k} = r_{0,k}\,\frac{1-\mu_0/\mu}{(1-2K_1\lambda^2)Q}, \qquad r_{j,k} = \frac{(2j+2k+1)!g_{j+k+1}}{(2j)!(2k+1)!2^{2j+2k+2}},$$

$$Q'_{0,k} = -r_{0,k}\,\frac{\mu_0/\mu}{(1-2K_1\lambda^2)Q}, \qquad Q_0 = \frac{\varepsilon - 1}{(1-2K_1\lambda^2)Q},$$

$$Q = -\frac{\mu_0}{\mu}\,\frac{1-\varepsilon_0}{2} + \frac{1+\varepsilon}{2} + \frac{1-\varepsilon}{1-2K_1\lambda^2}, \qquad g_{j+k+1} = {\sum_{m,n}}'\frac{1}{T^{2j+2k+2}},$$

$$\rho_j = {\sum_{m,n}}'\frac{\overline{T}}{T^{2j+1}}, \qquad T = \frac{1}{2}\,P_{mn},$$

$$d_{j,k} = \frac{(2j+1)\lambda^{2j+2k+2}S_{j,k}}{\gamma}, \qquad T_j = \frac{t_j}{\gamma}, \qquad S_{j,k} = \frac{1-\mu/\mu_0}{1-\varepsilon}\Big(\gamma_{j,k} + \frac{\mu_0}{\mu\varepsilon_0}\,\gamma^*_{j,k} + D_{j,k}\Big),$$

$$D_{j,k} = \frac{g_{j+1}g_{k+1}\lambda^2}{2^{2j+2k+4}}\,\eta\Big(\frac{\mu}{\mu_0}\Big), \qquad D_{0,0} = \lambda^2K_0^2\eta\Big(\frac{\mu}{\mu_0}\Big), \qquad \eta\Big(\frac{\mu}{\mu_0}\Big) = \frac{C}{C_1},$$

$$D_{0,k} = \lambda^2K_0^2\,\frac{g_{k+1}}{2^{2k+2}}\,\eta\Big(\frac{\mu}{\mu_0}\Big), \qquad T_j = T^*_j + H_j,$$

$$C = \frac{1+\varepsilon_0}{\varepsilon_0}\,\frac{1}{1-(1+\varepsilon)K_1\lambda^2} - \frac{2}{1-2K_1\lambda^2}, \qquad T^*_j = \frac{t^*_j}{\gamma},$$

$$C_1 = 1 - (1-2K_1\lambda^2)\Big(\frac{\mu}{\mu_0}\,\frac{1+\varepsilon_0}{1-\varepsilon} - \frac{1+\varepsilon}{1-\varepsilon}\Big), \qquad H_j = \frac{h_j}{\gamma},$$

$$t_0^* = \frac{M_y^\infty - M_x^\infty}{2(1-\nu)D}\Big(1 + \frac{\mu_0}{\varepsilon_0\mu}\Big), \qquad t_j^* = \frac{M_x^\infty + M_y^\infty}{4(1+\nu)D}\,\frac{(2j+1)g_{j+1}}{2^{2j+2}}\,\lambda^{2j+2}\eta_1\Big(\frac{\mu}{\mu_0}\Big),$$

$$\eta_1\Big(\frac{\mu}{\mu_0}\Big) = \frac{C_2}{C_3}, \qquad C_2 = \Big(1 - \frac{\mu_0}{\varepsilon_0\mu}\Big)\Big(1 + \varepsilon - \frac{\mu}{\mu_0}\,(1+\varepsilon_0)\Big),$$

$$C_3 = 1 - (1+\varepsilon)K_1\lambda^2 - \frac{\mu}{2\mu_0}\,(1+\varepsilon_0)(1 - 2K_1\lambda^2),$$

$$\gamma = \frac{(1-\mu/\mu_0)(1 - \varepsilon\mu_0/(\varepsilon_0\mu))}{1-\varepsilon} + \frac{1-\varepsilon_0}{\varepsilon_0}, \qquad h_0 = -\frac{1-\varepsilon_0}{\varepsilon_0}\sum_{k=0}^{\infty}\frac{g_{k+2}\lambda^{2k+4}}{2^{2k+4}}\,A_{-2k-2}^{\text{int}},$$

$$h_j = \frac{(2j+1)g_{j+1}\lambda^{2j+2}}{2^{2j+2}}\,(1-\varepsilon_0)\Big[\frac{\mu}{2Q\mu_0}\Big(\frac{1}{1-2K_1\lambda^2} - \frac{1+\varepsilon_0}{2(1-(1+\varepsilon)K_1\lambda^2)\varepsilon}\Big)$$

$$-\frac{1}{2(1-(1+\varepsilon)K_1\lambda^2)\varepsilon}\Big](-A_0^{\text{int}}) + \frac{1-\varepsilon_0}{\varepsilon_0}\sum_{k=0}^{\infty}\frac{(2j+2k+3)!\,g_{j+k+2}\lambda^{2j+2k+4}}{(2j)!(2k+3)!2^{2j+2k+4}}\,A_{2k+2}^{\text{int}}$$

(the prime at the sum indicates that the indices $m = n = 0$ are eliminated during summation).

In the case of regular meshes, which are of greatest interest for practice, system (8), (10), (11) can be simplified.

For a triangular mesh of holes [$\omega_1 = 2$ and $\omega_2 = 2\exp(i\pi/3)$], system (11) acquires the form

$$A_{6j} = \sum_{k=1}^{\infty} d_{3j-1,3k-1}A_{6k} + T_{3j-1} \qquad (j = 1, 2, \ldots).$$

For a quadratic mesh of holes ($\omega_1 = 2$ and $\omega_2 = 2i$), system (11) takes the form

$$A_{4j} = \sum_{k=1}^{\infty} d_{2j-1,2k-1}A_{4k} + T_{2j-1} \qquad (j = 1, 2, \ldots).$$

Systems (8), (10), and (11) obtained for a prescribed interference completely determine the solution of the problem of the stress–strain state of an elastic plate reinforced by disks made of another elastic material.

Until now, the interference was formally considered to be given. Let us return to the problem of optimization and find the coefficients A_{2k}^{int}. For this purpose, we construct the missing equations for closing systems (8), (10), and (11).

Using the formulas [11]

$$M_\theta + M_\rho = -4D(1+\nu)\operatorname{Re}\Phi(z),$$

$$M_\theta - M_\rho + 2iH_{\rho\theta} = 2D(1-\nu)[\bar{z}\Phi'(z) + \Psi(z)]\,\mathrm{e}^{2i\theta},$$

we find the bending moment M_θ on the contour of the hole in the plate $|\tau| = \lambda$:

$$M_\theta = \frac{1}{2}\,(M_x^\infty + M_y^\infty) + \frac{1}{2}\,(M_y^\infty - M_x^\infty)\cos 2\theta$$

$$-\,2D(1+v)\Big(\alpha_0 + \sum_{k=0}^{\infty}\alpha_{2k+2}\cos{(2k+2)\theta} + \sum_{k=0}^{\infty}\alpha_{2k+2}\sum_{j=0}^{\infty}\lambda^{2k+2j+2}r_{j,k}\cos 2j\theta\Big)$$

$$+\,D(1-\nu)\Big(\beta_0\cos 2\theta - \sum_{k=0}^{\infty}(2k+2)\alpha_{2k+2}\cos{(2k+2)\theta} + \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\alpha_{2k+2}\lambda^{2k+2j+2}2jr_{j,k}\cos 2j\theta$$

$$+\sum_{k=0}^{\infty}\beta_{2k+2}\cos 2k\theta + \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\beta_{2k+2}\lambda^{2k+2j+2}r_{j,k}\cos{(2j+2)\theta}$$

$$-\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(2k+2)\alpha_{2k+2}\lambda^{2k+2j+2}S_{j,k}\cos{(2j+2)\theta}\Big). \tag{12}$$

In Eq. (12), the coefficients α_{2k} and β_{2k} depend on A_{2k}^{int}, which are coefficients of the Fourier series of the sought interference function. To construct the missing equations that would allow determining these coefficients, we use the least squares technique, i.e., we choose the values of the coefficients A_{2k}^{int} that ensure minimization of stresses on the hole contour:

$$\sum_{i=1}^{N}[M_\theta(\theta_i) - M_0]^2 \to \min.$$

Here M_0 is the optimal value of the bending moment on the hole contour to be determined in solving the problem.

We divide the segment $[0, 2\pi]$ into N identical parts: $\Delta\theta = 2\pi/N$. At the division points (nodes) θ_i, we calculate the values of the function $M_\theta(\theta_i)$. The right side of the expression

$$U = \sum_{i=1}^{N}[M_\theta(\theta_i) - M_0]^2$$

is a function depending on the control parameters A_{2k}^{int} and M_0 and explicitly depending on the coefficients α_{2k} and β_{2k}. In turn, the coefficients α_{2k} and β_{2k} depend on the coefficients A_{2k}^{int} [see systems (8), (10), and (11)].

Using Eqs. (10), we eliminate the coefficients β_{2k} from Eqs. (12).

According to the least squares technique, the best coefficients $\alpha_{2k}(A_{2k}^{\text{int}})$ and M_0 are those that ensure the minimum values of the function U. Using the necessary condition of the extreme point of a function of several variables, we obtain an infinite system of equations for determining the values of M_0 and $\alpha_{2k}(A_{2k}^{\text{int}})$:

$$\frac{\partial U}{\partial M_0} = 0, \qquad \frac{\partial U}{\partial \alpha_{2k}} = 0 \qquad (k = 1, 2, \ldots). \tag{13}$$

System (13) is simplified because the function $M_\theta(\theta, \alpha_{2k})$ $(k = 1, 2, \ldots)$ is linear with respect to the parameters α_{2k} and can be presented in the form

$$M_\theta(\theta, \alpha_{2k}) = f_0 + \alpha_2 f_2(\theta) + \alpha_4 f_4(\theta) + \alpha_6 f_6(\theta) + \ldots + \alpha_{2k+2} f_{2k+2}(\theta) + \ldots\ .$$

The remaining quantities (partial derivatives) are readily found. With allowance for the relations obtained, we write the linear system of equations with respect to the unknowns $M_0, \alpha_2, \alpha_4, \ldots, \alpha_{2k+2}, \ldots$:

$$\begin{gathered}
-NM_0 + \alpha_0 \sum_{i=1}^{N} f_2(\theta_i) + \alpha_4 \sum_{i=1}^{N} f_4(\theta_i) + \ldots + \alpha_{2k+2} \sum_{i=1}^{N} f_{2k+2}(\theta_i) + \ldots = -\sum_{i=1}^{N} f_0(\theta_i), \\
\alpha_2(f_2, f_2) + \alpha_4(f_2, f_4) + \ldots + \alpha_{2k+2}(f_2, f_{2k+2}) + \ldots = (f_2, Y), \\
\alpha_2(f_4, f_2) + \alpha_4(f_4, f_4) + \ldots + \alpha_{2k+2}(f_4, f_{2k+2}) + \ldots = (f_4, Y), \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\alpha_2(f_{2k+2}, f_2) + \alpha_4(f_{2k+2}, f_4) + \ldots + \alpha_{2k+2}(f_{2k+2}, f_{2k+2}) + \ldots = (f_{2k+2}, Y), \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
(k = 0, 1, 2, \ldots).
\end{gathered} \tag{14}$$

In this system,

$$(f_j, f_k) = \sum_{i=1}^{N} f_j(\theta_i) f_k(\theta_i), \qquad (f_j, Y) = \sum_{i=1}^{N} f_j(\theta_i) Y_i, \qquad Y_i = M_0 - f_0(\theta_i).$$

Analysis of Results. The infinite system (14) together with systems (8), (10), and (11) allows one to determine the stress–strain state of the plate, the optimal interference for fitting elastic disks into the holes, and the optimal value of the normal tangential bending moment on the contour of the hole in the plate. These systems, however, are rather cumbersome. As $0 \leqslant \lambda < 1$, and the parameter λ in high powers enters these systems, the latter can be significantly simplified. In solving most of the practical problems, each system can be reduced to two or three equations; nevertheless, the results for the operating ranges of the hole radius λ will be fairly accurate.

TABLE 1

Calculated Coefficient A_{2k}^{int} for a Triangular Mesh of Holes

λ	A_0^{int}		A_6^{int}		A_{12}^{int}		A_{18}^{int}	
	OB	AB	OB	AB	OB	AB	OB	AB
0.2	0.071	0.062	0.043	0.034	0.019	0.011	0.006	0.003
0.3	0.094	0.078	0.062	0.048	0.023	0.017	0.009	0.005
0.4	0.117	0.102	0.078	0.069	0.034	0.021	0.012	0.009
0.5	0.136	0.122	0.080	0.071	0.041	0.032	0.015	0.011
0.6	0.159	0.141	0.095	0.084	0.053	0.039	0.019	0.013
0.7	0.172	0.158	0.092	0.086	0.066	0.055	0.027	0.021

Note. OB and AB refer to one-sided and all-sided bending, respectively.

TABLE 2

Calculated Coefficient A_{2k}^{int} for a Quadratic Mesh of Holes

λ	A_0^{int}		A_4^{int}		A_8^{int}		A_{12}^{int}	
	OB	AB	OB	AB	OB	AB	OB	AB
0.2	0.082	0.074	0.051	0.043	0.018	0.014	0.007	0.004
0.3	0.108	0.096	0.067	0.061	0.025	0.020	0.008	0.005
0.4	0.139	0.124	0.086	0.072	0.032	0.018	0.011	0.009
0.5	0.151	0.143	0.093	0.084	0.038	0.028	0.015	0.011
0.6	0.173	0.158	0.108	0.091	0.046	0.033	0.017	0.015
0.7	0.190	0.182	0.116	0.104	0.053	0.042	0.022	0.018

Note. OB and AB refer to one-sided and all-sided bending, respectively.

For the numerical implementation of the method described above, we solved the linear algebraic systems (8), (10), (11), and (14) by the method of truncation of algebraic systems. We examined one-sided bending of the plate by constant moments M_y^∞ ($M_x^\infty = 0$) and all-sided bending by the moments $M_x^\infty = M_y^\infty = M$ for regular meshes. The truncated systems were solved by the Gaussian method with the basic element chosen depending on the hole radius.

The coefficients A_{2k}^{int} calculated for different values of the hole radius are listed in Table 1 for a triangular mesh of holes and in Table 2 for a quadratic mesh of holes. The values used in calculations were $\nu = 0.30$ and $\mu = 2.5 \cdot 10^5$ MPa for the plate and $\nu_0 = 0.32$ and $\mu_0 = 3.6 \cdot 10^5$ MPa for the inclusion.

The case of annular disks is considered in a similar manner: the complex potentials $\Phi_0(z)$ and $\Psi_0(z)$ are sought in the form [11]

$$\Phi_0(z) = \sum_{k=-\infty}^{\infty} a_{2k} z^{2k}, \qquad \Psi_0(z) = \sum_{k=-\infty}^{\infty} a'_{2k} z^{2k}$$

and the boundary condition of the absence of forces on the inner contour of the annular disk is additionally imposed.

The problem with different interference criteria can also be considered in a similar manner.

It should be noted that the value of M_0 can be chosen in advance, on the basis of the load-carrying capacity of the plate. The calculations show, however, that the sum of squared deviations decreases in determining the unknown optimal values of M_θ, i.e., the search results are more accurate.

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